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On the relationship between classical Gaudin models and BC-type Gaudin models

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Abstract

Two invertible transformations of parameters and variables which connect classical $sl(2, \mathbb{C})$ Gaudin models with BC-type Gaudin models are proposed. The explicit corresponding formulae of the Lax matrices and the r-matrix relations are exhibited. As applications, we establish a relation of two kinds of restricted Kaup–Newell flows, present the BC-type counterpart of the well known restricted AKNS flow and obtain the Gaudin model corresponding to the Hikami system.

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1. Introduction

During the past decade there has been great progress in constructing finite-dimensional integrable Hamiltonian systems (FDIHSs). The method of nonlinearization of the Lax pair or the restricted flow technique [2, 6, 7, 16, 24] enables us to have obtained a large number of FDIHSs or so-called the restricted flows from (1 + 1)-dimensional soliton equations. These FDIHSs inherit many interesting integrable properties from soliton equations. For instance, all of them possess Lax representations [3, 23]. In the case of soliton equations connected with 2×2 matrix spectral problems, there is a wide class of the resulting FDIHSs whose Lax matrices are of the form [8, 17, 22, 23, 25]

$$L(\zeta) = A + \sum_{j=1}^{n} \frac{1}{\zeta - \zeta_j} \begin{pmatrix} Q_j P_j & -Q_j^2 \\ P_j^2 & -Q_j P_j \end{pmatrix} \equiv A + \Gamma \tag{1}$$

where A is a traceless matrix whose entries may depend on the dynamical variables $Q_1, \ldots, Q_n, P_1, \ldots, P_n$, the parameters $\zeta_1, \ldots, \zeta_n$ and the spectral parameter ζ. If the matrix A is a constant matrix, this kind of integrable models are widely known as the Gaudin magnets in the classical case, which have been extensively studied [4, 10, 13, 14, 21]. In the general case, integrable models determined by (1) are called reduced classical Gaudin models [12], which will now be referred to as just Gaudin models. Apart from Gaudin models, there are also a very limited number of other FDIHSs whose Lax matrices are of form [22, 26, 27]

$$\tilde{L}(\lambda) = \tilde{A} + \sum_{j=1}^{n} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix} \equiv \tilde{A} + \tilde{\Gamma} \tag{2}$$

where matrix $\tilde{A}$, whose entries may depend on $\lambda, \lambda_1, \ldots, \lambda_n$ and dynamical variables $q_1, \ldots, q_n, p_1, \ldots, p_n$, and thus $\tilde{L}(\lambda)$ belong to the twisted loop algebra

$$\widehat{sl(2, \mathbb{C})}[\lambda, \lambda^{-1}] = \left\{ X(\lambda) \in sl(2, \mathbb{C})[\lambda, \lambda^{-1}] | \sigma X(\lambda) \sigma = X(-\lambda), \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

i.e. an element $\xi(\lambda) \in \widehat{sl(2, \mathbb{C})}[\lambda, \lambda^{-1}]$ has the form

$$\xi(\lambda) = \sum_{i=-\infty}^{\infty} \xi_i \lambda^i \qquad \xi_i \in sl(2, \mathbb{C})$$

where ξ_i is a multiple of the diagonal matrix $\mathrm{diag}(1, -1)$ if i is even and ξ_i is off-diagonal if i is odd. As $\tilde{A} = 0$, these kind of integrable models were first introduced by Hikami and called as the classical BC-type Gaudin models or classical Gaudin models with boundary [11]. For the sake of brevity, for the general matrix $\tilde{A}$, we will also call the integrable models determined by (2) BC-type Gaudin models.

These two kinds of integrable models bear quite different features and apparently there are no relations between them. Algebraically, Gaudin models relate to loop algebra $sl(2, \mathbb{C})[\lambda, \lambda^{-1}]$, the semi-infinite Laurent series over $sl(2, \mathbb{C})$, while BC-type Gaudin models relate to the twisted loop algebra $\widehat{sl(2, \mathbb{C})}[\lambda, \lambda^{-1}]$ [1]. Since r-matrices correspond to special factorizations of associated algebras, these two kinds of integrable models possess essentially different r-matrices [18, 19]. This leads to the fact that integrating the former is easier than integrating the latter [11, 21, 26].

However, in this paper we shall show that, through introducing new parameters and dynamical variables, Gaudin models can be transformed into BC-type Gaudin models, and vice versa. Our starting point is the Lax matrix and its r-matrix relation. As is well known, the Lax matrix and its r-matrix relation contain almost all the necessary information for a FDIHS [5, 20]. The integrals of the system may be derived from the Lax matrix and their involutivity may be proved systematically with the help of r-matrix theory. In the next section, we first show that there are a series of invertible transformations which can transform Lax matrices of Gaudin models into those of BC-Gaudin models, and vice versa. Moreover, we also show that if the original Lax matrices satisfy r-matrix relations then the resulting Lax matrices also satisfy r-matrix relations. Thus, roughly speaking, we claim that Gaudin models can be transformed into BC-type Gaudin models, and vice versa. In order to obtain explicit relations between Gaudin models and BC-type Gaudin models, we make the basic assumption: suppose the Hamiltonian system with Hamiltonian H associates with its Lax matrix $L(u)$ satisfying an r-matrix relation in the following way. Set

$$\det L(u) = \sum_{j=k_0}^{\infty} F_j u^{-j}$$

where u is a spectral parameter and k_0 is a certain integer. Then, according to the general theory of r-matrix [5], F_k and F_j are in involution for any k and j. We assume that the Hamiltonian H is a smooth function of $F_{k_0}, \ldots, F_m$ up to a certain integer m. On this basis, we present two transformations of parameters and dynamical variables, which not only transform the Gaudin models (or Hamiltonian systems) into the BC-type Gaudin models, but also transform the

Hamiltonian functions as well as the conserved integrals of the motion of Gaudin models to that of the corresponding BC-type Gaudin models, and vice versa.

The results proposed here will allow us to generate new integrable Hamiltonian systems and reveal connections between integrable Hamiltonian systems. In section 3, a relation between two kinds of restricted Kaup–Newell flows is established, the BC-type counterpart of the well known restricted Ablowitz–Kaup–Newell–Segur (AKNS) flow is derived and the Gaudin model corresponding to the Hikami system is obtained. The paper closes with section 4, where some problems are pointed out.

2. Main results

The aim of this section is to give two transformations of parameters and dynamical variables which transform Gaudin models to BC-type Gaudin models and vice versa. Throughout this paper, with the exception of the special illustration, we always consider $Q_1, \ldots, Q_n$ and $P_1, \ldots, P_n$ are an n-canonically conjugate variable pair, as are $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$, namely

$$\{Q_i, P_j\} = \delta_{ij} \qquad \{Q_j, Q_k\} = \{P_j, P_k\} = 0$$
$$\{q_i, p_j\} = \delta_{ij} \qquad \{q_j, q_k\} = \{p_j, p_k\} = 0 \qquad i, j, k = 1, \ldots, n.$$

Usually we use the vector notation: $\boldsymbol{Q} = (Q_1, \ldots, Q_n)^T$, $\boldsymbol{P} = (P_1, \ldots, P_n)^T$, $\boldsymbol{q} = (q_1, \ldots, q_n)^T$, $\boldsymbol{p} = (p_1, \ldots, p_n)^T$ and use $\langle \cdot, \cdot \rangle$ to denote the standard inner product in $\mathbb{R}^n$. Also we assume $\Xi = \mathrm{diag}(\zeta_1, \ldots, \zeta_n)$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\zeta_1, \ldots, \zeta_n$ are n arbitrary given distinct nonzero constants, and so are $\lambda_1, \ldots, \lambda_n$.

We start from the classical Gaudin model with the Lax matrix

$$L(\zeta) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\zeta - \zeta_j} \begin{pmatrix} Q_j P_j & -Q_j^2 \\ P_j^2 & -Q_j P_j \end{pmatrix} \tag{3}$$

where a, b, c are some functions depending on the dynamical variables $Q_1, \ldots, Q_n$, $P_1, \ldots, P_n$, the parameters $\zeta_1, \ldots, \zeta_n$ and the spectral parameter ζ.

Let us analyse a series of transformations for the Lax matrix $L(\zeta)$ as follows. First, we introduce the following new parameters and variables

$$\lambda = \sqrt{\zeta} \qquad \lambda_j = \sqrt{\zeta_j} \qquad q_j = \lambda_j^{-\frac{1}{2}} Q_j \qquad p_j = \lambda_j^{\frac{1}{2}} P_j \qquad j = 1, 2, \ldots, n \tag{4}$$

and the Lax matrix (3) becomes

$$\tilde{L}^{(1)}(\lambda) = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & -\tilde{a} \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} q_j p_j & -\lambda_j q_j^2 \\ \lambda_j^{-1} p_j^2 & -q_j p_j \end{pmatrix} \tag{5}$$

with $\tilde{a} = a \circ \tau_1$, $\tilde{b} = b \circ \tau_1$, $\tilde{c} = c \circ \tau_1$, where

$$\tau_1 : \zeta \mapsto \lambda^2 \qquad \zeta_j \mapsto \lambda_j^2 \qquad Q_j \mapsto \lambda_j^{\frac{1}{2}} q_j \qquad P_j \mapsto \lambda_j^{-\frac{1}{2}} p_j \qquad j = 1, 2, \ldots, n$$

and symbol $\circ$ denotes the composition of functions, for instance

$$\begin{aligned} \tilde{a} &= a(\zeta, \zeta_1, \ldots, \zeta_n, Q_1, \ldots, Q_n, P_1, \ldots, P_n) \circ \tau_1 \\ &= a(\lambda^2, \lambda_1^2, \ldots, \lambda_n^2, \lambda_1^{\frac{1}{2}} q_1, \ldots, \lambda_n^{\frac{1}{2}} q_n, \lambda_1^{-\frac{1}{2}} p_1, \ldots, \lambda_n^{-\frac{1}{2}} p_n). \end{aligned}$$

Second, by multiplying $\tilde{L}^{(1)}(\lambda)$ with the parameter λ we get

$$\tilde{L}^{(2)}(\lambda) = \lambda \tilde{L}^{(1)}(\lambda) = \begin{pmatrix} \lambda\tilde{a} & \lambda\tilde{b} \\ \lambda\tilde{c} & -\lambda\tilde{a} \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda\lambda_j q_j^2 \\ \lambda\lambda_j^{-1} p_j^2 & -\lambda q_j p_j \end{pmatrix}. \tag{6}$$

Finally, making the similarity transformation

$$\tilde{L}(\lambda) = g^{-1}(\lambda)\tilde{L}^{(2)}(\lambda)g(\lambda) \qquad g(\lambda) = \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \tag{7}$$

thus we arrive at

$$\tilde{L}(\lambda) = \begin{pmatrix} \lambda\tilde{a} & \tilde{b} \\ \lambda^2\tilde{c} + \langle\Lambda^{-1}\boldsymbol{p}, \boldsymbol{p}\rangle & -\lambda\tilde{a} \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix}. \tag{8}$$

Since λ arises in $\tilde{a}, \tilde{b}, \tilde{c}$ in terms of λ^{2k} (k is an integer), we know that $\tilde{L}(\lambda) \in \widehat{sl(2,\mathbb{C})}[\lambda, \lambda^{-1}]$. Hence $\tilde{L}(\lambda)$ is of the form of the BC-type Gaudin model.

Now assume that the Lax matrix $L(\zeta)$ satisfies an r-matrix relation [5, 9]

$$\{L(\zeta) \stackrel{\otimes}{,} L(\nu)\} = [r_{12}(\zeta, \nu), L_1(\zeta)] + [r_{21}(\zeta, \nu), L_2(\nu)] \tag{9}$$

where we have used the standard notation [9]: $L_1(\zeta) = L(\zeta) \otimes I$, $L_2(\nu) = I \otimes L(\nu)$ and I is 2×2 unit matrix, ν is an arbitrary parameter. We shall show that $\tilde{L}(\lambda)$ also satisfies an r-matrix relation. To this end, we consider the above transformations one by one.

First, since the parameters $\zeta, \zeta_1, \ldots, \zeta_n$ in $L(\zeta)$ are arbitrary and the change of variables

$$q_j = \lambda_j^{-\frac{1}{2}} Q_j \qquad p_j = \lambda_j^{\frac{1}{2}} P_j \qquad j = 1, 2, \ldots, n$$

is a canonical transformation, the first transformation (4) changes the r-matrix relation to

$$\{\tilde{L}^{(1)}(\lambda) \stackrel{\otimes}{,} \tilde{L}^{(1)}(\mu)\} = [r_{12}^{(1)}(\lambda, \mu), \tilde{L}_1^{(1)}(\lambda)] + [r_{21}^{(1)}(\lambda, \mu), \tilde{L}_2^{(1)}(\mu)] \tag{10}$$

with the r-matrices $r_{12}^{(1)}(\lambda, \mu) = r_{12}(\zeta, \nu) \circ \tau_1$ and $r_{21}^{(1)}(\lambda, \mu) = r_{21}(\zeta, \nu) \circ \tau_1$.

Second, from the identity

$$\{\tilde{L}^{(2)}(\lambda) \stackrel{\otimes}{,} \tilde{L}^{(2)}(\mu)\} = [r_{12}^{(2)}(\lambda, \mu), \tilde{L}_1^{(2)}(\lambda)] + [r_{21}^{(2)}(\lambda, \mu), \tilde{L}_2^{(2)}(\mu)] \tag{11}$$

with $r_{12}^{(2)}(\lambda, \mu) = \mu r_{12}^{(1)}(\lambda, \mu), r_{21}^{(2)}(\lambda, \mu) = \lambda r_{21}^{(1)}(\lambda, \mu)$, it follows that we know $\tilde{L}^{(2)}(\lambda)$ satisfies an r-matrix relation.

Finally for the similarity transformation, a direct calculation shows that

$$\{\tilde{L}(\lambda) \stackrel{\otimes}{,} \tilde{L}(\mu)\} = [\tilde{r}_{12}(\lambda, \mu), \tilde{L}_1(\lambda)] + [\tilde{r}_{21}(\lambda, \mu), \tilde{L}_2(\mu)] \tag{12}$$

with

$$\tilde{r}_{12}(\lambda, \mu) = \frac{1}{\lambda}\mathrm{diag}(1, \mu, \lambda, \lambda\mu)(r_{12}(\zeta, \nu) \circ \tau_1)\mathrm{diag}(\lambda\mu, \lambda, \mu, 1)$$
$$\tilde{r}_{21}(\lambda, \mu) = \frac{1}{\mu}\mathrm{diag}(1, \mu, \lambda, \lambda\mu)(r_{21}(\zeta, \nu) \circ \tau_1)\mathrm{diag}(\lambda\mu, \lambda, \mu, 1).$$

This, together with the fact that all of these transformations are invertible, yields the following proposition.

Proposition 1. *There exist a series of invertible transformations which transform the Lax matrix of the Gaudin model*

$$L(\zeta) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\zeta - \zeta_j} \begin{pmatrix} Q_j P_j & -Q_j^2 \\ P_j^2 & -Q_j P_j \end{pmatrix} \tag{13}$$

into the Lax matrix of the BC-type Gaudin model

$$\tilde{L}(\lambda) = \begin{pmatrix} \lambda\tilde{a} & \tilde{b} \\ \lambda^2\tilde{c} + \langle\Lambda^{-1}\boldsymbol{p}, \boldsymbol{p}\rangle & -\lambda\tilde{a} \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix} \tag{14}$$

and vice versa. Furthermore, if the Lax matrix $L(\zeta)$ satisfies an r-matrix relation

$$\{L(\zeta) \overset{\otimes}{,} L(\nu)\} = [r_{12}(\zeta,\nu), L_1(\zeta)] + [r_{21}(\zeta,\nu), L_2(\nu)] \tag{15}$$

then $\tilde{L}(\lambda)$ satisfies the following r-matrix relation:

$$\{\tilde{L}(\lambda) \overset{\otimes}{,} \tilde{L}(\mu)\} = [\tilde{r}_{12}(\lambda,\mu), \tilde{L}_1(\lambda)] + [\tilde{r}_{21}(\lambda,\mu), \tilde{L}_2(\mu)] \tag{16}$$

with

$$\tilde{r}_{12}(\lambda,\mu) = \frac{1}{\lambda}\mathrm{diag}(1,\mu,\lambda,\lambda\mu)(r_{12}(\zeta,\nu)\circ\tau_1)\mathrm{diag}(\lambda\mu,\lambda,\mu,1)$$
$$\tilde{r}_{21}(\lambda,\mu) = \frac{1}{\mu}\mathrm{diag}(1,\mu,\lambda,\lambda\mu)(r_{21}(\zeta,\nu)\circ\tau_1)\mathrm{diag}(\lambda\mu,\lambda,\mu,1).$$

Remarks. It is not difficult to find that the connection between $\tilde{L}(\lambda) = \tilde{L}(\lambda,\lambda_1,\ldots,\lambda_n,q_1,\ldots,q_n,p_1,\ldots,p_n)$ and $L(\zeta) = L(\zeta,\zeta_1,\ldots,\zeta_n,Q_1,\ldots,Q_n,P_1,\ldots,P_n)$ can be simply expressed as

$$\tilde{L}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} (L(\zeta)\circ\tau_1) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

and

$$L(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta^{-\frac{1}{2}} \end{pmatrix} (\tilde{L}(\lambda)\circ\tilde{\tau}_1) \begin{pmatrix} \zeta^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

where $\tilde{\tau}_1$ is defined by

$$\tilde{\tau}_1 : \lambda \mapsto \zeta^{\frac{1}{2}} \qquad \lambda_j \mapsto \zeta_j^{\frac{1}{2}} \qquad q_j \mapsto \zeta_j^{-\frac{1}{4}} Q_j \qquad p_j \mapsto \zeta_j^{\frac{1}{4}} P_j \qquad j = 1,2,\ldots,n.$$

So far, we have shown how to transform a Lax pair of the Gaudin model into that of the BC-type Gaudin model. In what follows, we shall show that, through introducing new parameters and dynamical variables, we can directly transform Gaudin models into BC-type models. For this purpose, we first recall the construction of Hamiltonian systems from the Lax matrix satisfying an r-matrix. Suppose we have an integrable Gaudin model which has a Lax matrix $L(\zeta)$ and the Lax matrix satisfies an r-matrix relation. Furthermore let

$$\det L(\zeta) = \sum_{j=k_0}^{\infty} F_j \zeta^{-j}$$

where k_0 is certain integer. Then, by the hypothesis described in section 1, we choose the Hamiltonian function H to be a smooth function of $F_{k_0},\ldots,F_m$ up to a certain integer m. Thus all F_j's are the integrals of motion of the Hamiltonian flows with Hamiltonian H.

From (14), it is easy to see that

$$\tilde{L}(\lambda) = \begin{pmatrix} \lambda\tilde{a} & \tilde{b} \\ \lambda^2\tilde{c} & -\lambda\tilde{a} \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2-\lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda^2\lambda_j^{-1} p_j^2 & -\lambda q_j p_j \end{pmatrix} \tag{19}$$

thus

$$\det\tilde{L}(\lambda) = \lambda^2(\det L(\zeta)\circ\tau_1).$$

When expanding $\det\tilde{L}(\lambda)$ in terms of λ^{-1}, we have

$$\det\tilde{L}(\lambda) = \sum_{j=k_0}^{\infty} \tilde{F}_j \lambda^{-2j+2}$$

where

$$\tilde{F}_j = F_j \circ \tau_2 \tag{20}$$

and the map τ_2 is defined by

$$\tau_2 : \zeta_j \mapsto \lambda_j^2 \qquad Q_j \mapsto \lambda_j^{\frac{1}{2}} q_j \qquad P_j \mapsto \lambda_j^{-\frac{1}{2}} p_j \qquad j = 1, \ldots, n.$$

The above proposition tells us that $\tilde{L}(\lambda)$ satisfies an r-matrix relation and thus, for any j and k, $\tilde{F}_j$ and $\tilde{F}_k$ are still in involution. We may get the corresponding BC-type Gaudin model with Hamiltonian

$$\tilde{H} = H \circ \tau_2.$$

Hence, $\tilde{H}$ is also a smooth function of $\tilde{F}_{k_0}, \ldots, \tilde{F}_m$. This Hamiltonian system has a Lax matrix $\tilde{L}(\lambda)$ and has conserved integrals of motion $\tilde{F}_j$'s.

Theorem 1. *τ_2 transforms any Gaudin models determined by $L(\zeta)$ to their BC-type counterparts determined by $\tilde{L}(\lambda)$.*

Up to now, we have given a recipe to transform Gaudin models into BC-type Gaudin models. On the other hand, we notice the positions of Q_j, P_j, $(j = 1, 2, \ldots)$ are equal if we replace the first transformation with

$$\tau_3 : \zeta \mapsto \lambda^2 \qquad \zeta_j \to \lambda_j^2 \qquad Q_j \mapsto \lambda_j^{-\frac{1}{2}} q_j \qquad P_j \mapsto \lambda_j^{\frac{1}{2}} p_j \qquad j = 1, 2, \ldots, n$$

consequently, we get

$$\check{L}^{(2)}(\lambda) = \begin{pmatrix} \lambda\check{a} & \lambda\check{b} \\ \lambda\check{c} & -\lambda\check{a} \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda\lambda_j^{-1} q_j^2 \\ \lambda\lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix} \tag{21}$$

where $\check{a} = a \circ \tau_3$, $\check{b} = b \circ \tau_3$, $\check{c} = c \circ \tau_3$.

Furthermore, by replacing the similarity transformation with

$$\check{L}(\lambda) = g(\lambda)\check{L}^{(2)}(\lambda)g^{-1}(\lambda) \tag{22}$$

we arrive at

$$\check{L}(\lambda) = \begin{pmatrix} \lambda\check{a} & \lambda\check{b} - \langle \Lambda^{-1}\boldsymbol{q}, \boldsymbol{q}\rangle \\ \check{c} & -\lambda\check{a} \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix}. \tag{23}$$

Again, $\check{L}(\lambda)$ is of the form of the BC-type Gaudin model. In addition, we can prove that if the Lax matrix $L(\zeta)$ satisfies an r-matrix relation (9) then $\check{L}(\lambda)$ satisfies the following r-matrix relation

$$\{\check{L}(\lambda) \overset{\otimes}{,} \check{L}(\mu)\} = [\check{r}_{12}(\lambda, \mu), \check{L}_1(\lambda)] + [\check{r}_{21}(\lambda, \mu), \check{L}_2(\mu)] \tag{24}$$

with

$$\check{r}_{12}(\lambda, \mu) = \frac{1}{\lambda}\mathrm{diag}(\lambda\mu, \lambda, \mu, 1)(r_{12}(\zeta, \nu) \circ \tau_3)\mathrm{diag}(1, \mu, \lambda, \lambda\mu)$$

$$\check{r}_{21}(\lambda, \mu) = \frac{1}{\mu}\mathrm{diag}(\lambda\mu, \lambda, \mu, 1)(r_{21}(\zeta, \nu) \circ \tau_3)\mathrm{diag}(1, \mu, \lambda, \lambda\mu).$$

In a completely analogy analysis, we can show the following theorem.

Theorem 2. *τ_4 defined by*

$$\tau_4 : \zeta_j \mapsto \lambda_j^2 \qquad Q_j \mapsto \lambda_j^{-\frac{1}{2}} q_j \qquad P_j \mapsto \lambda_j^{\frac{1}{2}} p_j \qquad j = 1, \ldots, n$$

transforms Gaudin models into their BC-type counterparts determined by $\check{L}(\lambda)$.

3. Applications

3.1. Relations between two kinds of the restricted Kaup–Newell flows

In [7], starting from the spectral problem

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\zeta & \zeta u \\ v & \zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{25}$$

the following FDIHS was obtained

$$\begin{aligned} \boldsymbol{Q}_x &= -\Xi \boldsymbol{Q} - \langle \boldsymbol{Q}, \boldsymbol{Q}\rangle \Xi \boldsymbol{P} \equiv \frac{\partial H}{\partial \boldsymbol{P}} \\ \boldsymbol{P}_x &= \langle \Xi \boldsymbol{P}, \boldsymbol{P}\rangle \boldsymbol{Q} + \Xi \boldsymbol{P} \equiv -\frac{\partial H}{\partial \boldsymbol{Q}} \end{aligned} \tag{26}$$

where the Hamiltonian H is

$$H = -\langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle - \tfrac{1}{2}\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle\langle \Xi \boldsymbol{P}, \boldsymbol{P}\rangle.$$

This Hamiltonian system allows a Lax representation

$$L_x = [U, L] \tag{27}$$

with

$$L(\zeta) = \frac{1}{\zeta}\begin{pmatrix} -1 - \langle \boldsymbol{Q}, \boldsymbol{P}\rangle & 0 \\ -\langle \boldsymbol{P}, \boldsymbol{P}\rangle & 1 + \langle \boldsymbol{Q}, \boldsymbol{P}\rangle \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\zeta - \zeta_j}\begin{pmatrix} Q_j P_j & -Q_j^2 \\ P_j^2 & -Q_j P_j \end{pmatrix} \tag{28}$$

and

$$U = \begin{pmatrix} -\zeta & -\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle\zeta \\ \langle \Xi \boldsymbol{P}, \boldsymbol{P}\rangle & \zeta \end{pmatrix}.$$

We can check that the Lax matrix (28) satisfies the r-matrix relation (9) with

$$r_{12}(\zeta, \nu) = \frac{2}{\nu - \zeta} P + \frac{2}{\nu}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r_{21}(\zeta, \nu) = -P r_{12}(\nu, \zeta) P$$

where P is the standard permutation matrix [9]

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the Hamiltonian system (26) is a Gaudin model and $\det L(\zeta)$ is a generating function of its integrals of motion.

Writing down

$$\det L(\zeta) = \sum_{j=0}^{\infty} F_j \zeta^{-j-2}$$

then we have

$$F_0 = -1$$

$$F_1 = -2H$$

$$\begin{aligned} F_j = {} & 2\langle \Xi^j \boldsymbol{Q}, \boldsymbol{P}\rangle(1 + \langle \boldsymbol{Q}, \boldsymbol{P}\rangle) - \langle \boldsymbol{P}, \boldsymbol{P}\rangle\langle \Xi^j \boldsymbol{Q}, \boldsymbol{Q}\rangle \\ & + \sum_{k=0}^{j} \begin{vmatrix} \langle \Xi^k \boldsymbol{P}, \boldsymbol{P}\rangle & \langle \Xi^k \boldsymbol{Q}, \boldsymbol{P}\rangle \\ \langle \Xi^{j-k} \boldsymbol{Q}, \boldsymbol{P}\rangle & \langle \Xi^{j-k} \boldsymbol{Q}, \boldsymbol{Q}\rangle \end{vmatrix} \qquad j \geqslant 2. \end{aligned}$$

It has been shown that $F_1, \ldots, F_n$ are n independent conserved integrals of motion of (26). Since $H = -\frac{1}{2}F_1$, the Hamiltonian system (26) is completely integrable.

Following (14), we get the Lax matrix

$$\tilde{L}(\lambda) = \frac{1}{\lambda}\begin{pmatrix} -1-\langle \boldsymbol{q}, \boldsymbol{p}\rangle & 0 \\ 0 & 1+\langle \boldsymbol{q}, \boldsymbol{p}\rangle \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\lambda^2-\lambda_j^2}\begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix}. \tag{29}$$

By theorem 1, we know that the Lax matrix (29) satisfies the r-matrix relation (16) with

$$\tilde{r}_{12}(\lambda, \mu) = \frac{2}{\mu^2-\lambda^2}\begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} + \frac{2}{\mu}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

$$\tilde{r}_{21}(\lambda, \mu) = -P\tilde{r}_{12}(\mu, \lambda)P. \tag{31}$$

From (29) it follows that

$$\det \tilde{L}(\lambda) = \sum_{j=2}^{\infty} F_j \lambda^{-2j-2}$$

with

$$\begin{aligned} \tilde{F}_0 &= -1 \\ \tilde{F}_1 &= 2\langle \Lambda^2 \boldsymbol{q}, \boldsymbol{p}\rangle + \langle \Lambda \boldsymbol{q}, \boldsymbol{q}\rangle\langle \Lambda \boldsymbol{p}, \boldsymbol{p}\rangle \\ \tilde{F}_j &= 2\langle \Lambda^{2j} \boldsymbol{q}, \boldsymbol{p}\rangle(1+\langle \boldsymbol{q}, \boldsymbol{p}\rangle) - \langle \Lambda^{-1} \boldsymbol{p}, \boldsymbol{p}\rangle\langle \Lambda^{2j+1} \boldsymbol{p}, \boldsymbol{p}\rangle \\ &\quad + \sum_{k=1}^{j} \begin{vmatrix} \langle \Lambda^{2k+1} \boldsymbol{q}, \boldsymbol{q}\rangle & \langle \Lambda^{2k} \boldsymbol{q}, \boldsymbol{p}\rangle \\ \langle \Lambda^{2j-2k} \boldsymbol{q}, \boldsymbol{p}\rangle & \langle \Lambda^{2j-2k-1} \boldsymbol{p}, \boldsymbol{p}\rangle \end{vmatrix} \qquad j \geqslant 2. \end{aligned}$$

Choosing the Hamiltonian as $\tilde{H} = -\frac{1}{2}\tilde{F}_1$, we arrive at the following Gaudin-type integrable Hamiltonian system:

$$\begin{aligned} \boldsymbol{q}_x &= -\Lambda^2 \boldsymbol{q} - \langle \Lambda \boldsymbol{q}, \boldsymbol{q}\rangle \Lambda \boldsymbol{p} \equiv \frac{\partial \tilde{H}}{\partial \boldsymbol{p}} \\ \boldsymbol{p}_x &= \langle \Lambda \boldsymbol{p}, \boldsymbol{p}\rangle \Lambda \boldsymbol{q} + \Lambda^2 \boldsymbol{p} \equiv -\frac{\partial \tilde{H}}{\partial \boldsymbol{q}} \end{aligned} \tag{32}$$

which enjoys a Lax representation

$$\tilde{L}_x = [\tilde{U}, \tilde{L}] \tag{33}$$

where

$$\tilde{U} = \begin{pmatrix} -\lambda^2 & -\langle \Lambda \boldsymbol{Q}, \boldsymbol{Q}\rangle\lambda \\ \langle \Lambda \boldsymbol{P}, \boldsymbol{P}\rangle\lambda & \lambda^2 \end{pmatrix}. \tag{34}$$

The Hamiltonian system (32) has n independent conserved integrals of motion $\tilde{F}_1, \ldots, \tilde{F}_n$. It is not difficult to see that τ_2 maps the Hamiltonian system (26)–(32).

We notice that this Hamiltonian system can be obtained from another Kaup–Newell spectral problem

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda^2 & \lambda u \\ \lambda v & \lambda^2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{35}$$

through nonlinearization of the spectral problem. In [22] Zeng and Hietarinta have studied nonlinearization of this spectral problem and obtained an FDIHS which is slightly different from (32).

3.2. The BC-type counterpart of the restricted AKNS flows

The well known restricted AKNS flow is [7]

$$\begin{aligned} \boldsymbol{Q}_x &= -\Xi\boldsymbol{P} - \langle\boldsymbol{Q},\boldsymbol{Q}\rangle\boldsymbol{P} \\ \boldsymbol{P}_x &= \langle\boldsymbol{P},\boldsymbol{P}\rangle\boldsymbol{Q} + \Xi\boldsymbol{P} \end{aligned} \tag{36}$$

which can be put into a Hamiltonian system

$$\boldsymbol{Q}_x = \frac{\partial H}{\partial\boldsymbol{P}} \qquad \boldsymbol{P}_x = -\frac{\partial H}{\partial\boldsymbol{Q}}$$

with the Hamiltonian

$$H = -\langle\Xi\boldsymbol{Q},\boldsymbol{P}\rangle - \tfrac{1}{2}\langle\boldsymbol{P},\boldsymbol{P}\rangle\langle\boldsymbol{Q},\boldsymbol{Q}\rangle.$$

It allows a Lax representation

$$L_x(\zeta) = [U, L(\zeta)]$$

where

$$L(\zeta) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=1}^{N}\frac{1}{\zeta-\zeta_j}\begin{pmatrix} Q_jP_j & -Q_j^2 \\ P_j^2 & -Q_jP_j \end{pmatrix} \tag{37}$$

and

$$U = \begin{pmatrix} -\zeta & -\langle\boldsymbol{Q},\boldsymbol{Q}\rangle \\ \langle\boldsymbol{P},\boldsymbol{P}\rangle & \zeta \end{pmatrix}.$$

Therefore it is a Gaudin model. We can check that the Lax matrix $L(\zeta)$ satisfies the r-matrix relation (9) with

$$r_{12}(\zeta,\nu) = \frac{1}{\zeta-\nu}P \qquad r_{21}(\zeta,\nu) = -P\tilde{r}_{12}(\nu,\zeta)P.$$

Hence, $\det L(\zeta)$ is a generating function of integrals of motion of (36). Writing down

$$\det L(\zeta) = \sum_{j=0}^{\infty}F_j\zeta^{-j}$$

we have

$$\begin{aligned} F_0 &= -1 \\ F_1 &= 2\langle\Xi\boldsymbol{Q},\boldsymbol{P}\rangle + \langle\boldsymbol{Q},\boldsymbol{P}\rangle \\ F_j &= 2\langle\Xi^{j-1}\boldsymbol{Q},\boldsymbol{P}\rangle + \sum_{k=0}^{j-2}\begin{vmatrix} \langle\Xi^k\boldsymbol{P},\boldsymbol{P}\rangle & \langle\Xi^k\boldsymbol{Q},\boldsymbol{P}\rangle \\ \langle\Xi^{j-k}\boldsymbol{Q},\boldsymbol{P}\rangle & \langle\Xi^{j-k}\boldsymbol{Q},\boldsymbol{Q} \end{vmatrix} \qquad j \geqslant 2. \end{aligned}$$

In particular [15]

$$H = \tfrac{1}{2}F_2 + \tfrac{1}{8}F_1^2.$$

Now let us turn to construct its BC-type counterpart. The transformation τ_2 transforms the restricted AKNS flow (36) into the following finite-dimensional Hamiltonian system:

$$\begin{aligned} \boldsymbol{q}_x &= -\Lambda^2\boldsymbol{q} - \langle\Lambda\boldsymbol{q},\boldsymbol{q}\rangle\Lambda^{-1}\boldsymbol{p} = \frac{\partial\tilde{H}}{\partial\boldsymbol{p}} \\ \boldsymbol{p}_x &= \langle\Lambda^{-1}\boldsymbol{p},\boldsymbol{p}\rangle\Lambda\boldsymbol{q} + \Lambda^2\boldsymbol{p} = -\frac{\partial\tilde{H}}{\partial\boldsymbol{q}} \end{aligned} \tag{38}$$

with the Hamiltonian

$$\tilde{H} = H\circ\tau_2 = -\langle\Lambda^2\boldsymbol{q},\boldsymbol{p}\rangle - \tfrac{1}{2}\langle\Lambda^{-1}\boldsymbol{p},\boldsymbol{p}\rangle\langle\boldsymbol{q},\boldsymbol{q}\rangle.$$

This Hamiltonian system enjoys the following Lax representation:

$$\tilde{L}_x = [\tilde{U}, \tilde{L}]$$

where

$$\tilde{L}(\lambda) = \begin{pmatrix} -\lambda & 0 \\ \langle \Lambda^{-1}\boldsymbol{p}, \boldsymbol{p}\rangle & \lambda \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda p_j q_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix}$$

and

$$\tilde{U} = \begin{pmatrix} -\lambda^2 & -\langle \boldsymbol{q}, \boldsymbol{q}\rangle \lambda^{-1} \\ \langle \boldsymbol{p}, \boldsymbol{p}\rangle \lambda & \lambda^2 \end{pmatrix}.$$

Therefore, Hamiltonian system (38) is a BC-type Gaudin model.

By theorem 1 or by a direct check, we know that the Lax matrix $\tilde{L}(\lambda)$ satisfies the r-matrix relation (16) with

$$\tilde{r}_{12}(\lambda, \mu) = \frac{2}{\mu^2 - \lambda^2} \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} + \frac{2}{\lambda} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\tilde{r}_{21}(\lambda, \mu) = -P\tilde{r}_{12}(\mu, \lambda)P.$$

Thus $\det \tilde{L}(\lambda)$ is a generating function of the integrals of motion of (38). Writing down

$$\det \tilde{L}^2(\lambda) = \sum_{j=0}^{\infty} \tilde{F}_j \lambda^{2(1-j)}$$

we have

$$\begin{aligned}
\tilde{F}_0 &= -1 \\
\tilde{F}_1 &= 2\langle \boldsymbol{p}, \boldsymbol{q}\rangle \\
\tilde{F}_2 &= 2\langle \Lambda^2 \boldsymbol{p}, \boldsymbol{q}\rangle + \langle \Lambda^{-1}\boldsymbol{p}, \boldsymbol{p}\rangle\langle \Lambda\boldsymbol{q}, \boldsymbol{q}\rangle - \langle \boldsymbol{p}, \boldsymbol{q}\rangle^2 \\
\tilde{F}_j &= 2\langle \Lambda^{2j-2}\boldsymbol{p}, \boldsymbol{q}\rangle + \sum_{k=0}^{j-2} \begin{vmatrix} \langle \Lambda^{2k+1}\boldsymbol{q}, \boldsymbol{q}\rangle & \langle \Lambda^{2k}\boldsymbol{q}, \boldsymbol{p}\rangle \\ \langle \Lambda^{2j-2k}\boldsymbol{q}, \boldsymbol{p}\rangle & \langle \Lambda^{2j-2k-1}\boldsymbol{p}, \boldsymbol{p}\rangle \end{vmatrix} \qquad j > 2.
\end{aligned}$$

It is easy to see that

$$\tilde{H} = \tfrac{1}{2}\tilde{F}_2 + \tfrac{1}{8}\tilde{F}_1^2$$

and $\tilde{F}_1, \ldots, \tilde{F}_n$ are n independent conserved integrals of (38). Hamiltonian system (38) is a new completely integrable system.

3.3. The Gaudin model corresponding to the Hikami system

In this subsection, we shall derive the classical Gaudin model corresponding to the Hikami system. Also we shall show that our recipe can be applicable to not only the free Hamiltonian systems but also the Hamiltonian systems with constraints (or the so-called Neumann type Hamiltonian systems) to deduce the corresponding Gaudin model or BC-type Gaudin model.

Consider the simplest BC-type Gaudin model introduced by Hikami in [11]

$$\tilde{L}(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix} \equiv \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & -\tilde{A}(\lambda) \end{pmatrix}. \tag{39}$$

Hikami showed that this Lax matrix satisfies an r-matrix relation (9) in $\mathbb{R}^n$ with

$$\tilde{r}_{12}(\lambda,\mu)=\frac{2}{\mu^2-\lambda^2}\begin{pmatrix}\mu & 0 & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & 0 & \mu\end{pmatrix}\qquad \tilde{r}_{21}(\lambda,\mu)=-P\tilde{r}_{12}(\mu,\lambda)P.$$

Therefore, according to the general theory of r-matrices, $\tilde{L}(\lambda)$ generates a BC-type Gaudin model. On the other hand, in [27] we have proved that on the symplectic submanifold $\tilde{M}$ of $\mathbb{R}^{2n}$

$$\tilde{M}=\{(\boldsymbol{q},\boldsymbol{p})\in\mathbb{R}^{2n}|F\equiv\langle\Lambda\boldsymbol{p},\boldsymbol{p}\rangle-1=0, G\equiv\langle\boldsymbol{q},\boldsymbol{p}\rangle+\tfrac{1}{2}=0\}$$

under the Dirac bracket:

$$\{f,g\}_D=\{f,g\}+\tfrac{1}{2}(\{f,F\}\{g,G\}-\{f,G\}\{g,F\})$$

the Lax matrix (39) also admits the r-matrix representation

$$\{\tilde{L}(\lambda)\overset{\otimes}{,}\tilde{L}(\mu)\}_D=[\tilde{r}^D_{12}(\lambda,\mu),\tilde{L}_1(\lambda)]+[\tilde{r}^D_{21}(\lambda,\mu),\tilde{L}_2(\mu)]$$

with

$$\tilde{r}^D_{12}(\lambda,\mu)=\frac{2}{\mu^2-\lambda^2}\begin{pmatrix}\mu & 0 & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & 0 & \mu\end{pmatrix}+2S_1+2\lambda\tilde{S}_2$$

$$\tilde{r}^D_{21}(\lambda,\mu)=-P\tilde{r}^D_{12}(\mu,\lambda)P$$

where

$$S_1=\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\qquad \tilde{S}_2=\begin{pmatrix}0 & 0 & 0 & \tilde{B}(\mu)\\ 0 & 0 & -\tilde{C}(\lambda) & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{40}$$

Consequently, $\tilde{L}(\lambda)$ may generate integrable Hamiltonian systems on $\tilde{M}$, for example [27],

$$\begin{aligned}&\boldsymbol{q}_x=\Lambda^2\boldsymbol{q}-(\langle\Lambda^3\boldsymbol{p},\boldsymbol{p}\rangle+2\langle\Lambda^2\boldsymbol{q},\boldsymbol{p}\rangle)\boldsymbol{q}+2\langle\Lambda\boldsymbol{q},\boldsymbol{q}\rangle\Lambda\boldsymbol{p}\\ &\boldsymbol{p}_x=-\Lambda^2\boldsymbol{p}-2\Lambda\boldsymbol{q}+(\langle\Lambda^3\boldsymbol{p},\boldsymbol{p}\rangle+2\langle\Lambda^2\boldsymbol{q},\boldsymbol{p}\rangle)\boldsymbol{p}\\ &\langle\Lambda\boldsymbol{p},\boldsymbol{p}\rangle=1\qquad\langle\boldsymbol{q},\boldsymbol{p}\rangle=-\tfrac{1}{2}\end{aligned}\tag{41}$$

or in a Hamiltonian form

$$\begin{aligned}&\boldsymbol{q}_{j,x}=\{q_j,\tilde{H}\}_D\\ &\boldsymbol{p}_{j,x}=\{p_j,\tilde{H}\}_D\\ &\langle\Lambda\boldsymbol{p},\boldsymbol{p}\rangle=1\qquad\langle\boldsymbol{q},\boldsymbol{p}\rangle=-\tfrac{1}{2}\end{aligned}\tag{42}$$

where the Hamiltonian is

$$\tilde{H}=-\langle\Lambda^2\boldsymbol{q},\boldsymbol{p}\rangle-\tfrac{1}{2}\langle\Lambda^{-1}\boldsymbol{p},\boldsymbol{p}\rangle\langle\Lambda\boldsymbol{q},\boldsymbol{q}\rangle.$$

Denoting

$$\det\tilde{L}(\lambda)=\sum_{j=0}^{\infty}\tilde{F}_j\lambda^{-2j-2}$$

then we get

$$\tilde{F}_0=-\tfrac{1}{4}$$

$$\tilde{F}_1=\tilde{H}$$

$$\tilde{F}_j=-\langle\Lambda^{-1}\boldsymbol{p},\boldsymbol{p}\rangle\langle\Lambda^{2j+1}\boldsymbol{q},\boldsymbol{q}\rangle+\sum_{k=0}^{j}\begin{vmatrix}\langle\Lambda^{2k+1}\boldsymbol{q},\boldsymbol{q}\rangle & \langle\Lambda^{2k}\boldsymbol{q},\boldsymbol{p}\rangle\\ \langle\Lambda^{2j-2k}\boldsymbol{q},\boldsymbol{p}\rangle & \langle\Lambda^{2j-2k-1}\boldsymbol{p},\boldsymbol{p}\rangle\end{vmatrix}\qquad j\geqslant 1$$

where $\tilde{F}_1, \ldots, \tilde{F}_{n-1}$ are $n-1$ independent integrals of motion of (41) on submanifold $\tilde{M}$.

Now carrying out the inverse transformation procedures or by (18), we get a Lax matrix of the Gaudin models

$$L(\zeta) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\zeta}\langle \boldsymbol{P}, \boldsymbol{P}\rangle & 0 \end{pmatrix} + \sum_{j=1}^{n} \frac{1}{\zeta - \zeta_j} \begin{pmatrix} Q_j P_j & -Q_j^2 \\ P_j^2 & -Q_j P_j \end{pmatrix}$$
$$\equiv \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & -A(\zeta) \end{pmatrix}.$$

In $\mathbb{R}^n$, the Lax matrix (43) satisfies the r-matrix relation (9) with

$$r_{12}(\zeta, \nu) = \frac{2}{\nu - \zeta} P - \frac{2}{\nu} S \qquad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

On the submanifold M defined by

$$M = \{(\boldsymbol{Q}, \boldsymbol{P}) \in \mathbb{R}^{2n} \mid \langle \boldsymbol{P}, \boldsymbol{P}\rangle - 1 = 0, \langle \boldsymbol{Q}, \boldsymbol{P}\rangle + \tfrac{1}{2} = 0\}$$

The Lax matrix (43) satisfies the r-matrix relation

$$\{L(\zeta) \overset{\otimes}{,} L(\nu)\}_D = [r_{12}^D(\zeta, \nu), L_1(\zeta)] + [r_{21}^D(\zeta, \nu), L_2(\nu)]$$

with

$$r_{12}^D(\zeta, \nu) = \frac{2}{\nu - \zeta} P - \frac{2}{\nu} S + 2S_1 + 2\zeta S_2$$
$$r_{21}^D(\zeta, \nu) = -P r_{12}^D(\nu, \zeta) P$$

where S and S_1 are defined in (40) and (43) respectively, and

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & B(\nu) \\ 0 & 0 & -C(\zeta) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Corresponding to the Hamiltonian system (41), we get a new Hamiltonian system (a Gaudin model)

$$\begin{aligned} &\boldsymbol{Q}_x = \Xi \boldsymbol{Q} - (\langle \Xi^2 \boldsymbol{P}, \boldsymbol{P}\rangle + 2\langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle)\boldsymbol{Q} + 2\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle \Xi \boldsymbol{P} \\ &\boldsymbol{P}_x = -\Xi \boldsymbol{P} - 2\boldsymbol{Q} + (\langle \Xi^2 \boldsymbol{P}, \boldsymbol{P}\rangle + 2\langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle)\boldsymbol{P} \\ &\langle \Xi \boldsymbol{P}, \boldsymbol{P}\rangle = 1 \qquad \langle \boldsymbol{Q}, \boldsymbol{P}\rangle = -\tfrac{1}{2} \end{aligned} \tag{44}$$

or in Hamiltonian form

$$\begin{aligned} &Q_{j,x} = \{Q_j, H\}_D \\ &P_{j,x} = -\{P_j, H\}_D \\ &\langle \Xi \boldsymbol{P}, \boldsymbol{P}\rangle = 1 \qquad \langle \boldsymbol{Q}, \boldsymbol{P}\rangle = -\tfrac{1}{2} \end{aligned} \tag{45}$$

where the Hamiltonian H is

$$H = \langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle + \langle \boldsymbol{Q}, \boldsymbol{Q}\rangle.$$

This Hamiltonian system allows the following Lax representation:

$$L_x(\zeta) = [U, L]$$

where

$$U = \begin{pmatrix} \zeta - (\langle \Xi^2 \boldsymbol{P}, \boldsymbol{P}\rangle + 2\langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle) & 2\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle \zeta \\ -2 & -\zeta + \langle \Xi^2 \boldsymbol{P}, \boldsymbol{P}\rangle + 2\langle \Xi \boldsymbol{Q}, \boldsymbol{P}\rangle \end{pmatrix}.$$

The fact that $L(\zeta)$ satisfies an r-matrix together with

$$\det L(\zeta) = \sum_{j=0}^{\infty} F_j \zeta^{-j-2}$$

where

$$F_0 = -\tfrac{1}{4} \qquad F_1 = H$$
$$F_j = \langle \boldsymbol{P}, \boldsymbol{P}\rangle\langle \Xi^j \boldsymbol{Q}, \boldsymbol{Q}\rangle + \sum_{k=0}^{j} \begin{vmatrix} \langle \Xi^k \boldsymbol{Q}, \boldsymbol{Q}\rangle & \langle \Xi^k \boldsymbol{Q}, \boldsymbol{P}\rangle \\ \langle \Xi^{j-k} \boldsymbol{Q}, \boldsymbol{P}\rangle & \langle \Xi^{j-k} \boldsymbol{P}, \boldsymbol{P}\rangle \end{vmatrix} \qquad j \geqslant 2 \tag{46}$$

shows that $F_1, \ldots, F_{n-1}$ are $n-1$ conserved integrals of motion of (44). Moreover, they are functionally independent on M. Therefore (44) is a new completely integrable Hamiltonian system.

4. Conclusions and discussions

In summary, we have given two recipes to transform Gaudin models related to $sl(2, \mathbb{C})[\lambda, \lambda^{-1}]$ to BC-type Gaudin models related to the twisted loop algebra $\widehat{sl(2, \mathbb{C})}[\lambda, \lambda^{-1}]$, and vice versa. The corresponding formulae of Lax matrices and r-matrix relations have also been obtained. The recipes can not only be applied to free Hamiltonian systems but also can be applied to Hamiltonian systems with constraints. They provide an effective approach to construct new BC-type Gaudin models from Gaudin models. One natural and interesting problem is whether there exist other kinds of 'Gaudin models'? If so, what are the relations to the two kinds of integrable models discussed here? Another interesting problem is to consider the quantum integrable case.

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